TRANSACTIONS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 188, Issue 2, 1974

SYMMETRIC INTEGRO-DIFFERENTIAL-BOUNDARY PROBLEMS

BY

HYMAN J. ZIMMERBERG

ABSTRACT. Necessary and sufficient conditions for a linear vector integro-differential-boundary problem to be symmetric (selfadjoint) are developed, and then applied to obtain canonical forms of such symmetric problems. Moreover, the formulation of the integro-boundary conditions herein yields a simplification of one of the conditions for selfadjointness of a differential-boundary operator previously announced.

1. Introduction. Necessary and sufficient conditions for a class of differential-boundary operators to be selfadjoint, recently given by Krall [3], will be extended to integro-differential-boundary problems, in vector form,

$$A_1(x)y' + A_0(x)y + H(x)[M_2y(a) + N_2y(b)] + K(x) \int_a^b F(t)y\,dt = \lambda B(x)y, \tag{1.1a}$$

$$My(a) + Ny(b) + \int_a^b F(t)y\,dt = 0. \tag{1.1b}$$

In addition, for the special case $K(x) \equiv 0$ the formulation of the conditions (1.1b) herein allows a simplification of one of the conditions in [3].

A suitable formulation of a problem adjoint to a differential system with integral-boundary conditions has been the subject of a number of papers, among them [1], [2], [3] and [5]. However, the definition of symmetry (selfadjointness) adopted in this paper is more in accordance with that employed for boundary problems (see, for example, [4, p. 197]) than the requirement used in [3]. Moreover, while the integral term in (1.1a) may be replaced by a boundary term in view of condition (1.1b), nevertheless, there is an extra measure of generality in this setting that is unavailable in [3].

Presented to the Society, October 27, 1973; received by the editors March 27, 1973.

AMS (MOS) subject classifications (1970). Primary 34A30, 34B05, 47G05; Secondary 34A20, 47A05, 47A50, 47D99.

Key words and phrases. Adjoint, boundary value problem, canonical form, differential system, integro-boundary condition, integro-differential-boundary problem, linear, operator, selfadjoint, symmetric.

The adjoint problem will be devloped and necessary and sufficient conditions for symmetry will be derived in §2 for problems wherein the form of the integral-boundary condition (1.1b) is an extension of that used in [5]. Canonical forms for classes of equivalent symmetric integro-differential-boundary problems (1.1a), (1.1b) will then be constructed in §3.

Matrix and vector notation will be employed throughout. Matrices of various dimensional orders will be denoted by both italic and Greek capital letters, while vectors will be represented by lower-case italic letters. The operations of differentiation and conjugate-transpose for both matrices and vectors will be indicated by $'$ and $*$, respectively. Moreover, as is customary, 0 will be used indiscriminately to denote either the number zero, a zero matrix or a zero vector; the $\sigma \times \sigma$ identity matrix will be indicated by I_σ, and i will denote a complex square root of -1. Finally, when row and column dimensions agree, $[M, N; P, Q]$ will represent the matrix $\begin{bmatrix} M & N \\ P & Q \end{bmatrix}$.

2. **Necessary and sufficient conditions.** For the problem (1.1a), (1.1b) it will be assumed that the elements of the $n \times n$ matrix $A_1(x)$ are of class C' on the finite interval $a \leq x \leq b$, the elements of the $n \times n$ matrices $A_0(x)$ and $B(x)$, the $m \times n$ matrix $F(x)$, the $n \times m$ matrix $K(x)$, and the $n \times p$ matrix $H(x)$ are continuous on $[a, b]$, λ is a scalar parameter, M_2 and N_2 are each $p \times n$ constant matrices, and M and N are each $m \times n$ constant matrices, $0 \leq m \leq 2n$, such that the m boundary forms (1.1b) are linearly independent. A necessary and sufficient condition for the latter assumption is that the rows of the $m \times 3n$ matrix $[M\ N\ F(x)]$ are linearly independent on $a \leq x \leq b$ (Jones, [2, Theorem 2.1]).

Now, by a linear rearrangement of its rows, we may place the boundary conditions (1.1b) in the form

$$
\begin{aligned}
&M_0 y(a) + N_0 y(b) &&= 0,\\
&M_1 y(a) + N_1 y(b) + \int_a^b F_1(t) y\, dt &&= 0, \qquad (2.1b)\\
&\int_a^b F_2(t) y\, dt &&= 0,
\end{aligned}
$$

where M_0 and N_0 are each $\rho \times n$ constant matrices with $[M_0\ N_0]$ having rank ρ, M_1 and N_1 each $\sigma \times n$ matrices with $[M_1\ N_1]$ having rank σ, $F_1(x)$ and $F_2(x)$, respectively, $\sigma \times n$ and $\tau \times n$ matrices such that the $\sigma + \tau$ rows of $F_1(x)$ and $F_2(x)$ are a set of $\sigma + \tau$ linearly independent vectors on $[a, b]$, the $\rho + \sigma$ rows of $[M_0\ N_0]$ and $[M_1\ N_1]$ are linearly independent, and $\rho + \sigma + \tau = m$. Moreover, as this transformation may be effected by multiplying (1.1b) on the left by a suitable $m \times m$ nonsingular constant matrix D, the partitioning of $K(x)D^{-1} \equiv [K_0(x)\ K_1(x)\ K_2(x)]$ into ρ, σ and τ columns, respectively, in view of the last boundary condition of (2.1b), reduces (1.1a) to the form

$$A_1(x)y' + A_0(x)y + H(x)[M_2y(a) + N_2y(b)] + K_1(x) \int_a^b F_1(t)y\,dt = \lambda B(x)y. \tag{2.1a}$$

Furthermore, without loss of generality, we may assume that $p = 2n - (\rho + \sigma)$ and that the $2n \times 2n$ matrix

$$\begin{bmatrix} M_0 & N_0 \\ M_1 & N_1 \\ M_2 & N_2 \end{bmatrix} \tag{2.2}$$

is nonsingular (see, for example, Remark 6.2 of [5]).

Transforming the integro-differential-boundary problem (2.1a), (2.1b) into an equivalent two-point boundary problem in a manner analogous to that employed in [2], [3] and [5] yields the adjoint problem. Introducing new vectors

$$u_1 \equiv M_1y(a) + \int_a^x F_1(t)y\,dt, \qquad u_2 \equiv \int_a^x F_2(t)y\,dt,$$

$$s_1 \equiv u_1(b) - u_1(a) = \int_a^b F_1(t)y\,dt, \qquad s_2 \equiv M_2y(a) + N_2y(b),$$

problem (2.1a), (2.1b) is equivalent to the system consisting of $(3n + \sigma + \tau - \rho)$ linear differential equations and $2n + 2(\sigma + \tau)$ end-point conditions:

$$\begin{gathered} A_1(x)y' + A_0(x)y + K_1(x)s_1 + H(x)s_2 = \lambda B(x)y, \\ u_1' - F_1(x)y = 0, \quad u_2' - F_2(x)y = 0, \\ s_1' = 0, \quad s_2' = 0, \\ M_0y(a) + N_0y(b) = 0, \quad M_1y(a) - u_1(a) = 0, \quad N_1y(b) + u_1(b) = 0, \\ u_2(a) = 0, \quad u_2(b) = 0, \\ u_1(a) + s_1(a) - u_1(b) = 0, \quad M_2y(a) - s_2(a) + N_2y(b) = 0. \end{gathered} \tag{2.3}$$

Now, if the inverse of (2.2) is introduced and partitioned,

$$\begin{bmatrix} M_0 & N_0 \\ M_1 & N_1 \\ M_2 & N_2 \end{bmatrix} \begin{bmatrix} -P_0 & -P_1 & -P_2 \\ Q_0 & Q_1 & Q_2 \end{bmatrix} = I_{2n}, \tag{2.4}$$

where P_0 and Q_0 are each of dimension $n \times \rho$, P_1 and Q_1 each $n \times \sigma$ and P_2 and Q_2 each $n \times p$, then the system adjoint to (2.3) is readily obtained (see, for example, [4, §3.6]), being comprised of $3n + \sigma + \tau - \rho$ differential equations and $4n - 2\rho$ end-point conditions:

$$
(2.5)\quad
\begin{aligned}
-[A_1^*(x)z]' + A_0^*(x)z - F_1^*(x)v_1 - F_2^*(x)v_2 &= \lambda B^*(x)z,\\
v_1' = 0, \qquad v_2' &= 0,\\
-t_1' + K_1^*(x)z = 0, \qquad -t_2' + H^*(x)z &= 0,\\
P_2^*A_1^*(a)z(a) + Q_2^*A_1^*(b)z(b) + P_2^*M_1^*v_1(a) - Q_2^*N_1^*v_1(b) - t_2(a) &= 0,\\
P_1^*A_1^*(a)z(a) + Q_1^*A_1^*(b)z(b) + P_1^*M_1^*v_1(a) - Q_1^*N_1^*v_1(b) + t_1(a) &= 0,\\
t_1(b) = 0, \qquad t_2(b) &= 0.
\end{aligned}
$$

Then, eliminating v_1, t_1 and t_2 with the aid of relation (2.4), the problem

$$
(2.6a)\quad
\begin{aligned}
&-[A_1^*(x)z]' + A_0^*(x)z - F_1^*(x)[P_1^*A_1^*(a)z(a) + Q_1^*A_1^*(b)z(b)]\\
&\qquad + F_1^*(x)\int_a^b K_1^*(t)z\,dt - F_2^*(x)v_2 = \lambda B^*(x)z, \qquad v_2 = \text{constant},
\end{aligned}
$$

$$
(2.6b)\quad P_2^*A_1^*(a)z(a) + Q_2^*A_1^*(b)z(b) + \int_a^b H^*(t)z\,dt = 0
$$

will be defined as the integro-differential-boundary problem *adjoint* to problem (2.1a), (2.1b).

It is to be noted that, in general, the parameter v_2 cannot be eliminated from (2.6a). However, for the problems discussed by Krall [3] it was assumed that $\tau = 0$, and, hence, an adjoint problem not involving a parameter may be introduced. In addition, it is readily verifiable that the adjoint problem (2.6) is independent of a reformulation of system (2.1a), (2.1b) into an equivalent problem by the addition to (2.1a) of terms of the form

$$
\begin{aligned}
&J_0(x)[M_0y(a) + N_0y(b)]\\
&+ J_1(x)\left[M_1y(a) + N_1y(b) + \int_a^b F_1(t)y\,dt\right] + J_2(x)\int_a^b F_2(t)y\,dt,
\end{aligned}
$$

$J_i(x)$ $(i = 1, 2, 3)$ continuous on $[a, b]$. This invariance property of the adjoint will be applied in developing canonical forms in the next section.

A problem (2.1a), (2.1b) will be termed *symmetric* if the integral-boundary forms (2.1b) and (2.6b) are equivalent and the integro-differential-boundary operators in (2.1a) and (2.6a) coincide when applied to vectors of class C' satisfying the integro-boundary conditions (2.1b). This terminology is an extension of that applied to differential expressions in [4, p. 122] and corresponds to the definition of selfadjointness in [4, p. 197]. Focusing our attention first on the integral-boundary conditions, we have the following generalization of results contained in Theorem 5.1 of [3].

Lemma 2.1. *For a problem* (2.1a), (2.1b) *the integro-boundary forms* (2.1b) *and* (2.6b) *of the adjoint problem are equivalent if and only if*

(2.7)

(a) $\rho + \sigma = n$ *and* $\tau = 0$, *or equivalently,* $m = p = n$,

(b) $[M_0, N_0; M_1, N_1] \cdot \mathrm{diag}\{-A_1^{*-1}(a), A_1^{*-1}(b)\} \cdot [M_0, N_0; M_1, N_1]^* = 0$, *and*

(c) $H(x) \equiv F_1^*(x)E_1$ *on* $[a, b]$, *where* $E_1 \equiv [0\ I_\sigma]E$, $E \equiv C^{-1}$, *and*

$$C = [M_2 N_2] \cdot \mathrm{diag}\{-A_1^{-1}(a), A_1^{-1}(b)\} \cdot [M_0, N_0; M_1, N_1]^*.$$

As the rank of the $p \times 2n$ matrix $[P_2^* A_1^*(a)\ Q_2^* A_1^*(b)]$ is p, equivalence of the integro-boundary forms (2.1b) and (2.6b) requires that $m = p = \rho + \sigma$, whence (2.7a) follows. Then, under (2.7a) a necessary and sufficient condition for equivalence of the integro-boundary forms is the existence of an $n \times n$ constant nonsingular matrix E such that

$$[P_2^* A_1^*(a)\ Q_2^* A_1^*(b)] = E^*[M_0, N_0; M_1, N_1], \tag{2.8}$$

$$H^*(x) \equiv E^*[0\ F_1^*(x)]^* \quad \text{on } [a, b]. \tag{2.9}$$

Now, it follows from relation (2.4) that (2.7b) and (2.8) are equivalent with E as given in (2.7c). For this latter equivalence we first note that, as (2.7b) is an immediate consequence of (2.8), the nonsingularity of the matrix C, defined in (2.7c), follows either by a method analogous to that used in [3, p. 445] or, as (2.7b) is an immediate consequence of (2.8) for some nonsingular E, from the fact that the rows of $[M_2\ N_2]$ are linearly independent of the rows of $[M_0, N_0; M_1, N_1]$, while the latter rows constitute a maximal linearly independent vector set orthogonal to the $2n$ columns of $\mathrm{diag}\{-A_1^{-1}(a), A_1^{-1}(b)\} \cdot [M_0, N_0; M_1, N_1]^*$. Then, with E defined as in (2.7c), (2.8) follows from (2.7b) and the fact that the matrices P_2 and Q_2 are uniquely determined by the relations (2.4). Finally, (2.7c) and (2.9) are clearly equivalent under the definition of E_1 in (2.7c).

Lemma 2.2. *Under equivalence of the integro-boundary forms* (2.1b) *and* (2.6b), *problem* (2.1a), (2.1b) *is symmetric if and only if there exists a* $\sigma \times \sigma$ *constant Hermitian matrix* Γ *such that on* $a \leq x \leq b$

$$(2.10)\quad \begin{aligned} &\text{(a)}\quad A_1^*(x) \equiv -A_1(x),\; A_0^*(x) \equiv A_0(x) - A_1'(x),\; B^*(x) \equiv B(x), \\ &\text{(b)}\quad K_1(x) \equiv F_1^*(x)[\Gamma + \tfrac{1}{2}\Theta],\; \Theta \equiv P_1^* A_1(a) P_1 - Q_1^* A_1(b) Q_1. \end{aligned}$$

For a symmetric problem the relations in the first condition follow from term-by-term identification of corresponding operators when applied to vectors y of class C' satisfying $y(a) = y(b) = \int_a^b F_1(t) y\, dt = \int_a^b K_1^*(t) y\, dt = 0$. Then, under (2.7) and (2.10a), a necessary and sufficient condition for problem (2.1a), (2.1b) to be symmetric is that

$$(2.11)\quad \begin{aligned} &F_1^*(x)[(E_1 M_2 + P_1^* A_1^*(a)) y(a) + (E_1 N_2 + Q_1^* A_1^*(b)) y(b)] \\ &\qquad - F_1^*(x) \int_a^b K_1^*(t) y\, dt + K_1(x) \int_a^b F_1(t) y\, dt \equiv 0 \end{aligned}$$

on $[a, b]$ for arbitrary vectors y of class C' satisfying (2.1b), wherein $r = 0$. Now, for vectors $y \in C'$ such that $y(a) = y(b) = \int_a^b F_1(t) y\, dt = 0$, we have that $F_1^*(x) \int_a^b K_1^*(t) y\, dt = 0$; and, as the σ columns of $F_1^*(x)$ are linearly independent on $[a, b]$, $\int_a^b K_1^*(t) y\, dt = 0$. Consequently, it follows that $\int_a^b K_1^*(t) v\, dt = 0$ for arbitrary continuous vectors $v(t)$ satisfying $\int_a^b F_1(t) v\, dt = 0$. In addition, as $\int_a^b F_1(t) F_1^*(t)\, dt$ is nonsingular, let the $\sigma \times \sigma$ constant matrix Φ be determined by $\int_a^b F_1(t)[K_1(t) - F_1^*(t)\Phi]\, dt = 0$. Then, for $v(t) \equiv [K_1(t) - F_1^*(t)\Phi] c$, c an arbitrary σ-dimensional constant vector, we have that $\int_a^b [K_1^*(t) - \Phi^* F_1(t)] v(t)\, dt = 0$, and, consequently,

$$(2.12)\quad K_1(x) \equiv F_1^*(x)\Phi \quad \text{on } [a, b].$$

Moreover, under (2.12), in view of the linear independence of the columns of $F_1^*(x)$ on $[a, b]$, condition (2.11) for vectors $y \in C'$ satisfying (2.1b) with $r = 0$ reduces to the requirement that

$$(2.13)\quad (E_1 M_2 + P_1^* A_1^*(a) - \Theta M_1) y(a) + (E_1 N_2 + Q_1^* A_1^*(b) - \Theta N_1) y(b) = 0,$$

where $\Theta \equiv \Phi - \Phi^*$, for arbitrary vectors $y(a)$, $y(b)$ such that $M_0 y(a) + N_0 y(b) = 0$. Now, as the $n + \sigma$ columns of $[-P_1, -P_2; Q_1, Q_2]$ form a maximal set of linearly independent vectors orthogonal to the ρ rows of $[M_0\ N_0]$, relation (2.13) for $y(a)$, $y(b)$ satisfying $M_0 y(a) + N_0 y(b) = 0$ holds if and only if

$$\begin{aligned} &-(E_1M_2 + P_1^*A_1^*(a) - \Theta M_1)P_1 + (E_1N_2 + Q_1^*A_1^*(b) - \Theta N_1)Q_1 = 0, \\ &-(E_1M_2 + P_1^*A_1^*(a) - \Theta M_1)P_2 + (E_1N_2 + Q_1^*A_1^*(b) - \Theta N_1)Q_2 = 0; \end{aligned} \tag{2.14}$$

or, equivalently, if and only if

$$\Theta = -P_1^*A_1^*(a)P_1 + Q_1^*A_1^*(b)Q_1, \tag{2.15}$$

$$E_1 = P_1^*A_1^*(a)P_2 - Q_1^*A_1^*(b)Q_2. \tag{2.16}$$

However, (2.16) reduces to an identity under (2.8) and (2.10a). Consequently, on setting $\Gamma \equiv \tfrac{1}{2}(\Phi + \Phi^*)$ we have $\Phi = \Gamma + \tfrac{1}{2}\Theta$, and the lemma is established.

Moreover, as conditions (2.14) are equivalent to the existence of an $\sigma \times \rho$ constant matrix J such that

$$E_1M_2 + P_1^*A_1^*(a) - \Theta M_1 = JM_0, \qquad E_1N_2 + Q_1^*A_1^*(b) - \Theta N_1 = JN_0, \tag{2.17}$$

on substituting the values of P_1 and Q_1 from (2.17) into (2.15) it follows, from the definition of E_1 in (2.7c), that

$$\Theta = E_1(-M_2A_1^{*-1}(a)M_2^* + N_2A_1^{*-1}(b)N_2^*)E_1^*. \tag{2.18}$$

On the other hand, for $K_1(x)$ of the form (2.12) with $\Phi = \Gamma + \tfrac{1}{2}\Theta$, Γ a $\sigma \times \sigma$ constant Hermitian matrix and Θ given by (2.18), we have from (2.4) and (2.7b) that P_1 and Q_1 satisfy relations (2.17) with $J \equiv E_1(-M_2A_1^{*-1}(a)M_2^* + N_2A_1^{*-1}(b)N_2^*)E_0^*$, $E_0 \equiv [I_\rho \;\; 0]E$, and that Θ satisfies (2.15). Combining these facts with the two lemmas yields the following result.

Theorem 2.1. *The integro-differential-boundary problem* (2.1a), (2.1b) *is symmetric (selfadjoint) if and only if there exists a* $\sigma \times \sigma$ *constant Hermitian matrix* Γ *such that on* $a \leq x \leq b$

(2.19)

(a) $\rho + \sigma = n$ *and* $\tau = 0$,

(b) $[M_0, N_0; M_1, N_1] \cdot \operatorname{diag}\{-A_1^{-1}(a), A_1^{-1}(b)\} \cdot [M_0, N_0; M_1, N_1]^* = 0$,

(c) $A_1^*(x) \equiv -A_1(x)$, $A_0^*(x) \equiv A_0(x) - A_1'(x)$, $B^*(x) \equiv B(x)$,

(d) $H(x) \equiv F_1^*(x)E_1$,

(e) $K_1(x) \equiv F_1^*(x)[\Gamma + \tfrac{1}{2}\Theta]$, *where* E_1 *is given in* (2.7c) *and* Θ *in* (2.18).

For the special case $K_1(x) \equiv 0$ considered by Krall [3], condition (2.19e) reduces to $\Theta = 0$ in view of the linear independence of the columns of $F_1^*(x)$ on $[a, b]$. This simplification of the condition corresponding to condition (5.7) of Theorem 5.1 of [3] accrues from the above formulation of the integro-boundary conditions notwithstanding the more restrictive equivalence of terms under self-adjointness in [3].

Corollary. *An integro-differential-boundary problem* (2.1a), (2.1b) *with* $K_1(x) \equiv 0$ *on* $[a, b]$ *is symmetric if and only if relations* (2.19a, b, c, d) *hold on* $[a, b]$ *and* $E_1(M_2A_1^{-1}(a)M_2^* - N_2A^{-1}(b)N_2^*)E_1^* = 0$, E_1 *defined in* (2.7c).

3. Canonical forms. For a symmetric integro-differential-boundary problem

$$
\begin{aligned}
A_1(x)y' + A_0(x)y + H(x)[M_2y(a) + N_2y(b)] + K_1(x)\int_a^b F_1(t)y\,dt &= \lambda B(x)y,\\
M_0y(a) + N_0y(b) &= 0,\\
M_1y(a) + N_1y(b) + \int_a^b F_1(t)y\,dt &= 0,
\end{aligned}
\tag{3.1}
$$

with the $\rho \times 2n$ matrix $[M_0\ N_0]$, the $\sigma \times 2n$ matrix $[M_1\ N_1]$, the $n \times 2n$ matrix $[M_2\ N_2]$ and the $2n \times 2n$ matrix (2.2) each having maximal rank, and the σ rows of $F_1(x)$ linearly independent on $[a, b]$, it may be assumed, without loss of generality, that the orthogonality condition

$$M_0M_1^* + N_0N_1^* = 0 \tag{3.2}$$

holds. For, as the $\rho \times \rho$ matrix $W_0 \equiv M_0M_0^* + N_0N_0^*$ is nonsingular, the orthogonality relation (3.2) prevails upon adding to the last σ integro-boundary conditions of (3.1) the first ρ conditions premultiplied by $-(M_1M_0^* + N_1N_0^*)W_0^{-1}$; or equivalently, by premultiplying the integro-boundary conditions of (3.1) by the $n \times n$ nonsingular matrix

$$[I_\rho, 0;\ -(M_1M_0^* + N_1N_0^*)W_0^{-1}, I_\sigma]. \tag{3.3}$$

Moreover, inasmuch as the new integro-boundary conditions are equivalent to the original and the matrices $H(x)$, $K_1(x)$, $F_1(x)$, M_0, N_0, M_2, N_2, P_1, Q_1, P_2, Q_2, E_1 and Θ are unaffected by this change of M_1 and N_1, conditions (2.19) remain valid and the symmetry of the problem is preserved. It is to be noted that while the matrix C undergoes a postmultiplication by the conjugate-transpose of (3.3) the matrix E_1 remains unaltered. In this section *we shall assume that the orthogonality condition* (3.2) *holds.*

Now, as

$$-E_1M_2A_1^{-1}(a)M_0^* + E_1N_2A_1^{-1}(b)N_0^* = 0$$

from the definition of E_1 in (2.7c), and a maximal set of $2n-\rho$ linearly independent vectors of the null space of $[-M_0A_1^{*-1}(a)\ N_0A_1^{*-1}(b)]$ is given by the columns of the $2n \times (n+\sigma)$ matrix

$$\begin{bmatrix} M_0^* & M_1^* & -A_1^*(a)M_1^* \\ N_0^* & N_1^* & A_1^*(b)N_1^* \end{bmatrix}$$

in view of (2.19b) and (3.2), it follows that there exist $\sigma \times \rho$, $\sigma \times \sigma$ and $\sigma \times \sigma$ constant matrices R, S and T, respectively, such that

$$(3.4) \qquad E_1M_2 = RM_0 + SM_1 - TM_1A_1(a), \qquad E_1N_2 = RN_0 + SN_1 + TN_1A_1(b).$$

On postmultiplying these two relations by $-A_1^{-1}(a)M_1^*$ and $A_1^{-1}(b)N_1^*$, respectively, and adding, we have that $I_\sigma = TW_1$, where

$$(3.5) \qquad W_1 \equiv M_1M_1^* + N_1N_1^*$$

is a $\sigma \times \sigma$ nonsingular Hermitian matrix; and, hence, $T = W_1^{-1}$. Then, on postmultiplying relations (3.4), in turn, by M_1^*, N_1^* and $-A_1^{-1}(a)M_2^*E_1^*$, $A_1^{-1}(b)N_2^*E_1^*$, respectively, and adding, it follows from (2.18), (2.19b), (3.2) and the definition of E_1 that the matrix S has the representations:

$$S = \Omega + E_1(M_2M_1^* + N_2N_1^*)W_1^{-1}, \qquad S = \Theta + W_1^{-1}(M_1M_2^* + N_1N_2^*)E_1^*,$$

where

$$(3.6) \qquad \Omega \equiv W_1^{-1}(M_1A_1(a)M_1^* - N_1A_1(b)N_1^*)W_1^{-1}.$$

As $S - S^* = \Theta + \Omega$, the matrix $S - \frac{1}{2}\Theta - \frac{1}{2}\Omega$ is Hermitian; and, consequently,

$$H(x)[M_2 y(a) + N_2 y(b)] + K_1(x)\int_a^b F_1(t) y\, dt$$

$$= F_1^*(x)\{W_1^{-1}[-M_1A_1(a)y(a) + N_1A_1(b)y(b)] + \tfrac{1}{2}\Omega[M_1 y(a) + N_1 y(b)]\}$$

$$+ F_1^*(x)[\Gamma + \tfrac{1}{2}\Theta + \tfrac{1}{2}\Omega - S]\int_a^b F_1(t) y\, dt$$

for vectors $y \in C$ satisfying the integro-boundary conditions of (3.1). The invariance property of the adjoint problem, discussed in the previous section, then assures the following result.

Theorem 3.1. *Every symmetric integro-differential-boundary problem* (3.1) *is representable in the form*

$$(3.7)\quad \begin{aligned} A_1(x)y' + A_0(x)y + F_1^*(x)\{W_1^{-1}[-M_1A_1(a)y(a) + N_1A_1(b)y(b)] + \tfrac{1}{2}\Omega[M_1 y(a) + N_1 y(b)]\} \\ + F_1^*(x)\Psi\int_a^b F_1(t)y\, dt = \lambda B(x)y, \\ M_0 y(a) + N_0 y(b) = 0, \\ M_1 y(a) + N_1 y(b) + \int_a^b F_1(t) y\, dt = 0, \end{aligned}$$

where Ψ *is a* $\sigma \times \sigma$ *constant Hermitian matrix,* W_1 *is defined by* (3.5) *and* Ω *by* (3.6), *the orthogonality relations* (2.19b) *and* (3.2) *hold, and the* σ *rows of* $F_1(x)$ *are linearly independent and the matrices* $A_1(x)$, $A_0(x)$ *and* $B(x)$ *satisfy* (2.19c) *on* $[a, b]$.

Conversely, a problem (3.7) with the ρ rows of $[M_0\ N_0]$ and the σ rows of $[M_1\ N_1]$ constituting $\rho + \sigma = n$ linearly independent rows satisfying (2.19b) and (3.2), Ψ a $\sigma \times \sigma$ constant Hermitian matrix, W_1 and Ω given by (3.5) and (3.6), respectively, and for which relations (2.19c) hold and the σ rows of $F_1(x)$ are linearly independent on $[a, b]$, is symmetric and coincides with its adjoint problem term-by-term.

Moreover, for a symmetric problem (3.7) satisfying the conditions listed in Theorem 3.1 the rows of the end-point coefficient matrix $[M_0, N_0; M_1, N_1]$ may be orthonormalized, in view of condition (3.2), by premultiplying the two sets of integro-boundary conditions by the unique positive square roots of W_0^{-1} and W_1^{-1}, respectively. The resulting system may then be said to be in canonical form.

Corollary. *Every symmetric integro-differential-boundary problem* (3.1) *is reducible to the form*

$$
(3.8)\qquad
\begin{aligned}
&A_1(x)y' + A_0(x)y + F_1^*(x)[-M_1A_1(a)y(a) + N_1A_1(b)y(b)] \\
&\qquad\qquad + F_1^*(x)[\Psi + \Lambda] \int_a^b F_1(t)y\,dt = \lambda B(x)y, \\
&M_0y(a) + N_0y(b) = 0, \\
&M_1y(a) + N_1y(b) + \int_a^b F_1(t)y\,dt = 0,
\end{aligned}
$$

where $\Lambda \equiv (1/2)(-M_1A_1(a)M_1^* + N_1A_1(b)N_1^*)$, Ψ *is a* $\sigma \times \sigma$ *constant Hermitian matrix, relations* (2.19c) *hold and the* σ *rows of* $F_1(x)$ *are linearly independent on* $[a, b]$, *the rows of the* $n \times 2n$ *matrix* $[M_0, N_0; M_1, N_1]$ *are orthonormalized in the sense that* $M_0M_0^* + N_0N_0^* = I_\rho$, $M_1M_1^* + N_1N_1^* = I_\sigma$ *and* $M_0M_1^* + N_0N_1^* = 0$, *and*

$$[M_0, N_0; M_1, N_1] \cdot \mathrm{diag}\{-A^{-1}(a), A^{-1}(b)\} \cdot [M_0, N_0; M_1, N_1]^* = 0.$$

In particular, if $\Lambda = 0$ for a symmetric problem (3.1), as it does, for example, if $A_1(x)$ is a constant multiple of a unitary matrix, the forms (3.7) and (3.8) are further reduced.

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DEPARTMENT OF MATHEMATICS, HILL CENTER FOR THE MATHEMATICAL SCIENCES, RUTGERS UNIVERSITY, NEW BRUNSWICK, NEW JERSEY 08903